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ON SYMMETRIC FINSLER SPACES*

BY

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ABSTRACT

In this paper, we study symmetric Finsler spaces. We first study some geometric properties of globally symmetric Finsler spaces and prove that any such space can be written as a coset space of a Lie group with an invariant Finsler metric. Then we prove that a globally symmetric Finsler space is a Berwald space. As an application, we use the notion of Minkowski symmetric Lie algebras to give an algebraic description of symmetric Finsler spaces and obtain the formulas for flag curvature and Ricci scalar. Finally, some rigidity results of locally symmetric Finsler spaces related to the flag curvature are also given.

Introduction

The study of Finsler geometry has become active recently due to the excellent works of many geometers. In particular, the publication of several substantial books has attracted more and more people to this interesting field (cf. [4], [7], [15], etc.) One of the important motivations to study Finsler geometry is that it has important applications in Physics and Biology ([2]). We must also note that recently D. Bao, C. Robles and Z. Shen used the Randers metrics in Finsler geometry to study Zermelo navigation on Riemannian manifolds ([5]). However, only little attention has been paid to the study of symmetry of such spaces (see, for example, [12]). As a contrary comparison, the theory of Riemannian

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symmetric spaces, mainly due to É. Cartan, plays a very important role in Riemannian geometry.

The purpose of this paper is to study the symmetry of Finsler spaces. A locally symmetric Finsler space is, by definition, a Finsler space (M, F) such that for any $x \in M$, there exists a neighborhood N_x of x such that the geodesic symmetry S_x with respect to x is a local isometry of N_x ([12]). It is obvious that such a space must be reversible. Following É. Cartan's definition, we call a Finsler space (M, F) globally symmetric if each point of M is the isolated fixed point of an involutive isometry.

Here we must give some remarks on isometries of a Finsler space. Let (M, F) be a Finsler space, where F is positively homogeneous of degree one (but perhaps not absolutely homogeneous). Then we have two ways to define the notion of an isometry of (M, F). On the one hand, we call a diffeomorphism σ of M onto itself an isometry if $F(d\sigma_x(y)) = F(y)$, for any $x \in M$ and $y \in T_x(M)$. On the other hand, we can also define an isometry of (M, F) to be a one-to-one mapping of M onto itself which preserves the distance of each (ordered) pair of points of M. It is well-known (cf. [13], for example) that the two definitions are equivalent if the metric F is Riemannian. In [8], we proved that this is also true for a Finsler metric. Using this result, we proved that the group of isometries $I(M, F)$ of a Finsler space (M, F) is a Lie transformation group of M and for any point $x \in M$, the isotropic subgroup $I_x(M, F)$ is a compact subgroup of $I(M, F)$. These results are very important to study symmetric Finsler spaces.

In this paper, we study locally and globally symmetric Finsler spaces. After studying some general geometrical properties, we prove that a globally symmetric Finsler space (M, F) can be written as $(G/H, F)$, where G/H is a coset space and F is a G-invariant Finsler metric on G/H. Moreover, (G, H) is an effective Riemannian symmetric pair. Using this result, we prove that a globally symmetric Finsler space is a Berwald space. Then we use the notion of Minkowski symmetric Lie algebras to give an algebraic description of globally symmetric Finsler spaces. In particular, we study the duality, the decomposition theorem, the flag curvature and the Ricci scalar of symmetric Finsler spaces. Finally, we study locally symmetric Finsler spaces and obtain a geometric description of complete locally symmetric Finsler spaces. As an application of the results in this paper, we obtain some global rigidity results in Finsler manifolds, which, in some sense, generalize some results of Foulon ([12]).

The arrangement of this paper is as follows: in Section 1, we present some preliminaries on Finsler geometry. In particular, we introduce the Chern connection and the definitions of flag curvature and Ricci scalar. In Section 2, we study the general geometric properties of a globally symmetric Finsler space and prove that each such space must be Berwaldian. In Sections 3 through 7, we introduce the notion of a Minkwoski symmetric Lie algebra to give an algebraic description of symmetric Finsler spaces and study the duality, curvature and decomposition theorems. In Section 8, we obtain a geometric description of complete locally symmetric Finsler spaces and present some rigidity results. Finally, we make a conjecture to conclude this paper.

It should be noted that some of the results in Sections 3 through 6 of this paper overlap with the results of our previous paper, in which we consider globally symmetric Berwald spaces. However, for the convenience of the readers and for the completeness of this paper, we state the details of these results. But we omit the proof, which we refer to [11].

Notation: We use Einstein's abbreviated notation of summation: Any repeated pair of indices—provided that one is up and the other is down—is automatically summed.

1. Preliminaries

In this section, we recall some definitions and fundamental results in Finsler geometry. In particular, we will introduce the Chern connection, which is a useful tool to study the geometric properties of Finsler spaces.

1.1 Finsler spaces.

Definition 1.1: Let V be a n-dimensional real vector space. A Minkowski norm on V is a functional F on V which is smooth on $V-\{0\}$ and satisfies the following conditions:

(1) $F(u) \geq 0$, $\forall u \in V$;
(2) $F(\lambda u) = \lambda F(u)$, $\forall \lambda > 0$;
(3) for any basis $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ of V, write $F(y) = F(y^1, y^2, \ldots, y^n)$ for $y = y^j \varepsilon_j$. Then the Hessian matrix

$$(g_{ij}) := \left(\left[\frac{1}{2} F^2 \right]_{y^i y^j} \right)$$

is positive-definite at any point of $V - \{0\}$.

Example: Let $\langle\ ,\ \rangle$ be an inner product on V. Define $F(y) = \sqrt{\langle y, y\rangle}$. Then F is a Minkowski norm. In this case it is called Euclidean or coming from an inner product.

It can be shown ([4]) that for a Minkowski norm F, we have $F(u) > 0$, $\forall u \neq 0$. Furthermore

$$F(u_1 + u_2) \leq F(u_1) + F(u_2),$$

where the equality holds if and only if $u_2 = \alpha u_1$ or $u_1 = \alpha u_2$ for some $\alpha \geq 0$.

For any Minkowski norm F on real vector space V we define

$$C_{ijk} = \frac{1}{4}[F^2]_{y^i y^j y^k}.$$

Then for any $y \neq 0$, we can define two tensors on V, namely,

$$g_y(u, v) = g_{ij}(y) u^i v^j,$$
$$C_y(u, v, w) = C_{ijk}(y) u^i v^j w^k.$$

They are called the fundamental form and the Cartan torsion, respectively.

Definition 1.2: Let M be a (connected) smooth manifold. A Finsler metric on M is a function F: $TM \to [0, \infty)$ such that

(1) F is C^∞ on the slit tangent bundle $TM - \{0\}$;
(2) the restriction of F to any T_xM, $x \in M$ is a Minkowski norm.

Let (M, F) be a Finsler space and $x, y \in M$. For any smooth curve $\sigma(t)$, $0 \leq t \leq 1$ connecting x and y, we can define the length of the curve by

$$L(\sigma) = \int_0^1 F(\sigma(t), \sigma'(t)) dt.$$

Similarly, we can define the length of any piece-wise smooth curve connecting x and y. The distance function d of (M, F) is defined by

$$d(x, y) = \inf_{\sigma \in \Gamma(x,y)} L(\sigma),$$

where $\Gamma(x, y)$ denotes the set of all piece-wise smooth curves emanating from x to y. It can be proved ([4]) that $d(x, y) \geq 0$ with the equality holds if and only if $x = y$. Moreover, $d(x, y) \leq d(x, z) + d(z, y)$, $\forall x, y, z \in M$. However, generically we cannot have $d(x, y) = d(y, x)$. Therefore, d is not a distance in the general sense.

1.2 The Chern connection. Let (M, F) be a Finsler space and

$$(x^1, x^2, \ldots, x^n)$$

be a local coordinate system on an open subset U of M. Then $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ form a basis for the tangent space at any point in U. For $y \in T_x(M)$, $x \in U$, write $y = y^j \frac{\partial}{\partial x^j}$. Then $(x^1, x^2, \ldots, x^n, y^1, y^2, \ldots, y^n)$ is a (standard) coordinate system on TU. Using the coefficients g_{ij} and C_{ijk}, we define

$$C^i{}_{jk} = g^{is} C_{sjk},$$

where (g^{ij}) is the inverse matrix of (g_{ij}). The formal Christofell symbols of the second kind are

$$\gamma^i{}_{jk} = g^{is} \frac{1}{2} \left(\frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{ks}}{\partial x^j} \right).$$

They are functions on $TU - \{0\}$. We can also define some other quantities on $TU - \{0\}$ by

$$N^i{}_j(x, y) := \gamma^i{}_{jk} y^k - C^i{}_{jk} \gamma^k{}_{rs} y^r y^s,$$

where $y = y^i \frac{\partial}{\partial x^i} \in T_x(M) - \{0\}$.

Now the slit tangent bundle $TM - \{0\}$ is a fibre bundle over the manifold M with the natural projection π. Since TM is a vector bundle over M, we have a pull-back bundle $\pi^* TM$ over $TM - \{0\}$.

Theorem 1.1 ([3]): *The pull-back bundle $\pi^* TM$ admits a unique linear connection, called the Chern connection, which is torsion free and almost g-compatible. The coefficients of the connection in the standard coordinate system is*

$$\Gamma^l{}_{jk} = \gamma^l{}_{jk} - g^{li} \Big(A_{ijs} \frac{N^s{}_k}{F} - A_{jks} \frac{N^s{}_i}{F} + A_{kis} \frac{N^s{}_j}{F} \Big).$$

1.3 The flag curvature and the Ricci scalar. Let (M, F) be a Finsler space, $(x^1, x^2, \ldots, x^n)$ be a local coordinate system and Γ^i_{jk} be the coefficients of the Chern connection. Define $\omega^i_j = \Gamma^i_{jk} dx^k$. To define the flag curvature, we need some differential forms on the manifold $TM - \{0\}$. Let

$$\delta y^i = dy^i + N^i{}_j dx^j.$$

The curvature 2-forms of the Chern connection are

$$\Omega^i{}_j = d\omega^i{}_j - \omega^k{}_j \wedge \omega^i{}_k.$$

Since $\Omega^i{}_j$ are 2-forms on the manifold $TM - \{0\}$, they can be expanded as

$$\Omega^i{}_j = \frac{1}{2} R_j{}^i{}_{kl} dx^k \wedge dx^l + P_j{}^i{}_{kl} dx^k \wedge \frac{\delta y^l}{F} + \frac{1}{2} Q_j{}^i{}_{kl} \frac{\delta y^k}{F} \wedge \frac{\delta y^l}{F}.$$

(It turns out that ${Q_j}^i{}_{kl} = 0$.) Let

$$R_{jikl} = g_{is} {R_j}^s{}_{kl}.$$

Now we can define the notion of flag curvature. A flag on M at $x \in M$ is a pair (P, y), where P is a plane in the tangent space T_xM and y is a nonzero vector in P. The flag curvature of the flag (P, y) is defined to be

$$K(P, y) := \frac{u^i(y^j R_{jikl} y^l)u^k}{g_y(y, y)g_y(u, u) - [g_y(y, u)]^2},$$

where $u = u^i \frac{\partial}{\partial x^i}$ is any nonzero vector in P such that $P = \mathrm{span}\{y, u\}$. It can be shown that the quantity is independent of the selection of u ([4]). The Ricci scalar is defined as follows. For $x \in M$ and $y \in T_xM - \{0\}$, let $l = \frac{y}{F(y)}$ (the distinguished section). Then select $n-1$ vectors in $T_x(M)$, say $v_1, v_2, \ldots, v_{n-1}$, such that $l, v_1, v_2, \ldots, v_{n-1}$ form an orthonormal basis of $T_x(M)$ with respect to the inner product $g_l(\cdot, \cdot)$. Let $P_i = \mathrm{span}(y, v_i)$, $i = 1, 2, \ldots, n-1$. Then the Ricci scalar at y is defined to be

$$\mathbf{Ric}(y) = \sum_{i=1}^{n-1} K(P_i, y).$$

It can be shown that $\mathbf{Ric}(y)$ is equal to the trace of the endomorphism $u \to R_y(u)$ of the vector space $T_x(M)$ (see [7] for the details).

2. Globally symmetric Finsler spaces

The definition of globally symmetric Finsler space is a natural generalization of É. Cartan's definition of Riemannian globally symmetric spaces. We call a Finsler space (M, F) a globally symmetric Finsler space if for any point $x \in M$ there exists an involutive isometry σ_x (that is, $\sigma_x^2 = \mathrm{id}$ but $\sigma_x \neq \mathrm{id}$) of (M, F) such that x is an isolated fixed point of σ_x. We first give an effective method to construct globally symmetric Finsler spaces. In this paper, manifolds are always assumed to be connected.

First recall the notion of symmetric coset spaces. Let G be a Lie group, K a closed subgroup of G. Then the coset space G/K is called symmetric if there exists an involutive automorphism σ of G such that

$$G_\sigma^0 \subset K \subset G_\sigma,$$

where G_σ is the subgroup consisting of the fixed points of σ in G and G_σ^0 denotes the identity component of G_σ.

THEOREM 2.1: *Let G/K be a symmetric coset space. Then any G-invariant reversible Finsler metric (if exists) F on G/K makes $(G/K, F)$ a globally symmetric Finsler space.*

Proof: We first define some diffeomorphisms of G/K onto itself. Let $o = eK$ be the origin of G/K. Define a mapping σ_o of G/K onto itself by

$$\sigma_o(aK) = \sigma(a)K, \quad a \in G.$$

Then for any $x \in G/K$, we select an arbitrary $a \in G$ such that $x = \pi(a)$, where π is the natural projection of G onto G/K. Define a mapping σ_x of G/K onto itself by

$$\sigma_x = \tau_a \sigma_o \tau_a^{-1},$$

where τ_a is defined by $\tau_a : gK \to agK$, $g \in G$. By the definition of symmetric coset spaces, it is easily seen that σ_x is independent of the choice of a. It is well-known that σ_x is an involutive diffeomorphism of G/K with x as an isolated fixed point (cf. [14]). Next we prove that it is an isometry. Since F is G-invariant, τ_a keeps F invariant, $\forall a \in G$. Therefore, we only need to prove the case of σ_o. Let $\mathfrak{g}$, $\mathfrak{k}$ be the algebra of G, K, respectively. Then σ induces an involutive automorphism (still denote by σ) of $\mathfrak{g}$. By the definition, $\mathfrak{k}$ coincides with the set of fixed points of σ. Let $\mathfrak{m}$ be the eigenspace of σ with eigenvalue -1. Then we have

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m} \quad \text{(direct sum of subspaces)}.$$

Therefore, we can identify the tangent space $T_o(G/K)$ with $\mathfrak{m}$. This way, F corresponds to a norm on $\mathfrak{m}$ which is K-invariant. By the results of [8] (see the remarks about isometries in the introduction), we only need to check that $F(\sigma(y)) = F(y)$, $\forall y \in \mathfrak{m}$, i.e., $F(y) = F(-y)$, $\forall y \in \mathfrak{m}$. But this is obvious because F is reversible. ∎

Using the above theorem, we can construct a large number of globally symmetric Finsler metrics which is non-Riemannian.

Example 1: Let G_1/K_1, G_2/K_2 be two symmetric coset spaces with K_1, K_2 compact (in this case, they are Riemannian symmetric spaces) and g_1, g_2 be invariant Riemannian metric on G_1/K_1, G_2/K_2, respectively. Let $M = G_1/K_1 \times G_2/K_2$ and o_1, o_2 be the origin of G_1/K_1, G_2/K_2, respectively, and denote $o = (o_1, o_2)$ (the origin of M). Now, for

$$y = y_1 + y_2 \in T_o(M) = T_{o_1}(G_1/K_1) \dot{+} T_{o_2}(G_2/K_2),$$

we define

$$F(y) = \sqrt{g_1(y_1, y_1) + g_2(y_2, y_2) + \sqrt[s]{g_1(y_1, y_1)^s + g_2(y_2, y_2)^s}},$$

where s is any integer ≥ 2. Then $F(y)$ is a Minkowski norm on $T_o(M)$ which is invariant under $K_1 \times K_2$. Hence it defines an G-invariant Finsler metric on M (cf. [9]). By Theorem 1.1, (M, F) is a globally symmetric Finsler space. It is easy to check that F is non-Riemannian.

Next we consider the reverse of Theorem 1.1. For this purpose, we first need to study the geometric properties of globally symmetric Finsler spaces.

THEOREM 2.2: *Let (M, F) be a globally symmetric Finsler space. For $x \in M$, denote the involutive isometry of (M, F) at x by σ_x. Then we have*

(a) *For any $x \in M$, $(d\sigma_x)_x = -id$. In particular, F must be reversible;*
(b) *(M, F) is (forward and backward) complete;*
(c) *(M, F) is homogeneous. That is, the group of isometries of (M, F), $I(M, F)$, acts transitively on M.*
(d) *Let $\tilde{M}$ be the universal covering space of M and π be the projection mapping. Then $(\tilde{M}, \pi^*(F))$ is a globally symmetric Finsler space, where $\pi^*(F)$ is define by*

$$\pi^*(F)(y) = F((d\pi)_{\tilde{x}}(y)), \quad y \in T_{\tilde{x}}(\tilde{M}).$$

Proof: (a) It is known ([4]) that there exists a neighborhood U of the origin of $T_x(M)$ such that the exponential mapping $\exp_x$ is a (C^1)-diffeomorphism of U onto its image and for any $u \in U$, $\exp_x(tu)$, $t < |\varepsilon|$ is a geodesic through x. Since σ_x is an isometry, and in a Finsler space short geodesics are minimizing ([4]), we easily see that σ_x maps geodesics into geodesics. Now $\sigma_x(\exp_x(tu)$ and $\exp_x((d\sigma_x)_x tu)$ are two geodesics through x with the same initial vector $(d\sigma_x)_x u$. Therefore, they coincide as geodesics. In particular, we have $\sigma_x(\exp_x(u)) = \exp_x((d\sigma_x)_x u)$. Since $\sigma_x^2 = \mathrm{id}$, we have $(d\sigma_x)_x^2 = \mathrm{id}$. To prove $(d\sigma_x)_x = -\mathrm{id}$, we only need to prove that the number 1 is not an eigenvalue of $(d\sigma_x)_x$. Suppose, conversely, that there exists $u \neq 0$ such that $(d\sigma_x)_x(u) = u$. Then $(d\sigma_x)_x(tu) = tu$, $t \in \mathbb{R}$. Therefore, for any t we have

$$\sigma_x(\exp_x(tu)) = \exp_x((d\sigma_x)_x)(tu) = \exp_x(tu).$$

But this contradicts the assumption that x is an isolated fixed point of σ_x. Therefore, we have $(d\sigma_x)_x = -\mathrm{id}$.

(b) Let $\gamma(t)$, $0 \le t \le l$ be a geodesic parametrized to have constant Finslerian speed. We construct a curve $\tilde{\gamma}$ by $\tilde{\gamma}(t) = \gamma(t)$, for $0 \le t \le l$; $\tilde{\gamma}(t) = \sigma_{\gamma(l)}(\gamma(2l-t))$, for $l \le t \le 2l$. Similarly as in (a), we see that $\sigma_{\gamma(l)}(\gamma(2l-t))$ is a geodesic. By (a), the incoming vector of the geodesic γ at $\gamma(l)$ coincides the vector of the geodesic $\tilde{\gamma}$ at $\gamma(l)$. Therefore, by the uniqueness of the geodesics ([4]), $\tilde{\gamma}(t)$ is smooth and $\tilde{\gamma}(t)$, $0 \le t \le 2l$ is a geodesic. It is obvious that this geodesic still has constant Finslerian speed (since $\sigma_{\gamma(l)}$ is an isometry). Therefore, (M, F) is forward geodesically complete (cf. [4]). By the Hopf-Rinow theorem for Finsler spaces (cf. [4]), (M, F) is forward complete. Since F is reversible, (M, F) is complete.

(c) Since (M, F) is complete, for any $x, y \in M$, there exists a unit speed minimal geodesic $\gamma(t)$, $0 \le t \le T$ which realizes the distance of each pair of points in γ (cf. [4]). Let $x_0 = \gamma(T/2)$. Note that F is reversible ((a)), we have $d(x_0, x) = d(x, x_0) = d(x_0, y)$. By the proof of (a), we know that $\sigma_{x_0}(x) = y$. Therefore, (M, F) is homogeneous.

d) Let (M, F) be a globally symmetric Finsler space. Then it is easily seen that $(\tilde{M}, \pi^*(F))$ is a Finsler space. It is well-known that any diffeomorphism σ of M can be lifted to a diffeomorphism $\tilde{\sigma}$ of $\tilde{M}$ such that $\sigma\pi = \pi\tilde{\sigma}$. Furthermore, if $\sigma(x) = x$, then for any $\tilde{x} \in \tilde{\pi}^{-1}(x)$, we can take $\tilde{\sigma}$ such that $\tilde{\sigma}(\tilde{x}) = \tilde{x}$. Now for any $\tilde{y} \in \tilde{M}$, denote $y = \pi(\tilde{y})$. Then there exists a diffeomorphism $\tilde{\sigma}_{\tilde{y}}$ of $\tilde{M}$ such that $\sigma_y\pi = \pi\tilde{\sigma}_{\tilde{y}}$ and $\tilde{\sigma}_{\tilde{y}}(\tilde{y}) = \tilde{y}$. Since $\sigma_x^2 = \mathrm{id}_M$, we have $\tilde{\sigma}_{\tilde{y}}^2\pi^{-1}(y) = \pi^{-1}(y)$, $\forall y \in M$. Since $\tilde{\sigma}_{\tilde{y}}$ is a diffeomorphism keeping $\tilde{y}$ fixed, we see that $\tilde{\sigma}_{\tilde{y}}^2 = \mathrm{id}_{\tilde{M}}$. It is obvious that $\tilde{y}$ is an isolated fixed point of $\tilde{\sigma}_{\tilde{y}}$. By the definition of $\pi^*(F)$, we see that $\tilde{\sigma}_{\tilde{y}}$ is an isometry. Therefore, $(\tilde{M}, \pi^*(F))$ is a globally symmetric Finsler space. ∎

Corollary 2.3: *Let (M, F) be a globally symmetric Finsler space. Then for any $x \in M$, σ_x is a local geodesic symmetry at x. In particular, (M, F) is locally symmetric and for any $x \in M$, the symmetry σ_x is unique.*

Proof: By (a) of Theorem 2.2, we know that for any $x \in M$, $(d\sigma_x)_x = -\mathrm{id}$. Therefore, for any $X \in T_x(M)$, we have

$$\sigma_x(\exp_x(tX)) = \exp_x((d\sigma_x)_x(tX)) = \exp_x(-tX). \tag{2.1}$$

This means that σ_x is the local geodesic symmetry at x. Hence (M, F) is locally symmetric. Since (M, F) is complete (Theorem 2.2 (b)), $\exp_x$ is defined on the whole space $T_x(M)$ and it is surjective (cf. [4]). Therefore, by (2.1), σ_x is unique. ∎

THEOREM 2.4: *Let (M, F) be a globally symmetric Finsler space. Then there exits a Riemannian symmetric pair (G, K) such that M is diffeomorphic to G/K and F is invariant under G.*

Proof: By (c) of Theorem 2.2, the group $I(M, F)$ of isometries of (M, F) acts transitively on M. In [8], we proved that $I(M, F)$ is a Lie transformation group of M and for any $x \in M$, the isotropic subgroup $I_x(M, F)$ is a compact subgroup of $I(M, F)$. Since M is connected, the identity component $G = I^0(M, F)$ of $I(M, F)$ is also transitive on M ([13]) and the subgroup K of G which leaves x fixed is a compact subgroup of G. Furthermore, M is diffeomorphic to G/K under the mapping $gH \to g \cdot x$, $g \in G$.

Similarly as in the Riemannian case, we define a mapping σ of G into itself by: $\sigma(g) = \sigma_x g \sigma_x$, where σ_x denote the (unique) involutive isometry of (M, F) with x as an isolated fixed point. Then it is easily seen that σ is an involutive automorphism of G and the group K lies between the closed subgroup K_σ of fixed points of σ and the identity component of K_σ. Furthermore, the group K contains no normal subgroup of G other than $\{e\}$. That is, (G, K) is a symmetric pair. Since K is compact, (G, K) is a Riemannian symmetric pair (cf. [13]). ∎

Now we can prove an important result. Before the proof, let us first recall some definitions and known results. A Finsler space (M, F) is called a Berwald space if in any standard coordinate system $(x^1, x^2, \ldots, x^n, y^1, y^2, \ldots, y^n)$ the coefficients of the Chern connection, $\Gamma^i_{jk}(x, y)$, have no independence on the vector y, or in other words, if the Chern connection defines a linear connection directly on the underlying manifold. One should refer to [4] for detailed properties of Berwald spaces. To prove the following theorem, we still need the notion of parallel translations and holonomy groups in Finsler geometry. Now we give a sketch of them (see [7] for details).

Let (M, F) be a Finsler space. Γ^i_{jk} be the Coefficients of the Chern connection in some standard coordinate system. Let N^i_j be as in Section 1 (it turns out that $N^i_k = y^m \Gamma^i_{mk}$). Let c be a piecewise C^∞ curve in M. For a vector filed $U = U^i(t) \frac{\partial}{\partial x^i}|_{c(t)}$, define the (nonlinear) covariant derivative $D_{\dot{c}} U(t)$ of U along c by

$$D_{\dot{c}} U(t) = \{U^i(t) + \dot{c}^j(t) N^i_j(c(t), U(t))\} \frac{\partial}{\partial x^i}\Big|_{c(t)}.$$

U is called parallel along c if $D_{\dot{c}} U(t) = 0$. Define $P_c : T_p(M) \to T_q(M)$ by

$$P_c(u) = U(b),$$

here $U = U(t)$ denotes the (unique) parallel vector field along c with $U(a) = u$. P_c is called the parallel translation along c. It is easily seen that P_c is a C^∞ diffeomorphism from $T_pM - \{0\}$ onto $T_qM - \{0\}$, which is positively homogeneous of degree one, i.e., $P_c(\lambda u) = \lambda P_c(u), \lambda > 0$, $u \in T_p(M)$.

Using the notion of parallel translation, we can define the holonomy group of (M, F) at a point $p \in M$, denoted by H_p, similarly as in the Riemannian case (cf. [14]). The group H_p consists of diffeomorphisms of $T_p(M) - \{0\}$. Since $\forall \sigma \in H_p$, $F(\sigma(u)) = F(u)$, $u \in T_p(M)$, H_p is a transformation group of the indicatrix

$$\mathcal{I}_p = \{y \in T_p(M) \colon F(y) = 1\}.$$

Two Finsler metrics F, $\bar{F}$ are called affinely equivalent if they have the same geodesics as parametrized curves, that is, if $\sigma(t)$ is a geodesic of F, then it is also a geodesic of $\bar{F}$ and vice versa. The following result gives a method to use holonomy group to construct Finsler metrics which is affinely equivalent to a given Finsler metric.

Proposition 2.5 ([7]): *Let $(M, \bar{F})$ be a Finsler space, $p \in M$ and H_p be the holonomy group of $\bar{F}$ at p. If F_p is a H_p invariant Minkowski norm on $T_p(M)$, then F_p can be extended to a Finsler metric F on M by parallel translations of $\bar{F}$ such that F is affinely equivalent to $\bar{F}$.*

We also need the following

Proposition 2.6 ([7]): *A Finsler metric F on a manifold M is a Berwald metric if and only if it is affinely equivalent to a Riemannian metric g. In this case, F and g have the same holonomy group at any point $p \in M$.*

Now we can prove

Theorem 2.7: *Let (M, F) be a globally symmetric Finsler space. Then (M, F) is a Berwald space. Furthermore, the connection of F coincides with the Levi-Civita connection of a Riemannian metric g such that (M, g) is a Riemannian globally symmetric space.*

Proof: We first prove that F is Berwaldian. By Theorem 2.4, there exists a Riemannian symmetric pair (G, K) such that M is diffeomorphic to G/K and F is invariant under G. Fix a G-invariant Riemmannian metric g on G/K. Without loss of generality, we can assume that (G, K) is effective. Since being a Berwald space is a local property, we can assume further that G/K is simply connected (see Theorem 2.2, (d)). Then we have a decomposition ([13]):

$$G/K = E \times G_1/K_1 \times G_2/K_2 \times \cdots \times G_n/K_n,$$

where E is a Euclidean space, G_i/K_i are simply connected irreducible Riemannian globally symmetric spaces, $i = 1, 2, \ldots, n$. Now we determine the holonomy group of g at the origin of G/K. According to the de Rham decomposition theorem (cf. [14]), it is equal to the product of the holonomy groups of E and G_i/K_i at the origin. Now E has trivial holonomy group. For G_i/K_i, by the holonomy theorem of Ambrose and Singer ([1]), we know that the Lie algebra $\mathfrak{h}_i$ of the holonomy group H_i is spanned by the linear mappings of the form $\{\tilde{\tau}^{-1} R_o(X, Y)\tilde{\tau}\}$, where τ denotes any piecewise smooth curve starting from o, $\tilde{\tau}$ denotes parallel displacements (with respect to the restricted Riemannian metric) along $\tilde{\tau}$, $\tilde{\tau}^{-1}$ is the inverse of $\tilde{\tau}$, R_o is the curvature tensor of G_i/K_i of the restricted Riemannian metric and $X, Y \in T_o(G_i/K_i)$. Since G_i/K_i is a globally Riemannian symmetric space, the curvature tensor is invariant under parallel displacements ([13]). Therefore,

$$\mathfrak{h}_i = \mathrm{span}\{R_o(X, Y) |\, X, Y \in T_o(G_i/K_i)\}.$$

On the other hand, Since G_i is a semisimple group, we know that the Lie algebra of $K_i^* = Ad(K_i) \simeq K$ is also equal to the span of $R_o(X, Y)$ ([13], p. 207). Since G_i/K_i is simply connected, the groups H_i and K_i^* are connected ([13] and [14]). Therefore, we have $H_i = K_i^*$. Consequently the holonomy group H_o of G/K at the origin is

$$K_1^* \times K_2^* \times \cdots \times K_n^*.$$

Now F defines a Minkwoski norm F_o on $T_o(G/K)$ which is invariant by H_o. By Proposition 2.5, we can construct a Finsler metric $\bar{F}$ on G/K by parallel translations of g. By Proposition 2.6, $\bar{F}$ is Berwaldian. Now, for any point $p_0 = aK \in G/K$, there exists a geodesic of the Riemannian manifold $(G/K, g)$, say $\gamma(t)$, such that $\gamma(0) = o$, $\gamma(1) = p_0$. Suppose the initial vector of γ is X_0 and take $X \in \mathfrak{p}$ such that $d\pi(X) = X_0$. Then it is known that $\gamma(t) = \exp tX \cdot p_0$ and $d\tau(\exp tX)$ is the parallel translate of $(G/K, g)$ along γ ([13]). Since F is G-invariant, it is invariant under this parallel translate. This means that F and $\bar{F}$ coincide at $T_{p_0}(G/K)$. Consequently they coincide everywhere. Thus F is a Berwald metric.

For the next assertion ,we use a result of Szabó ([16]) which asserts that for any Berwald metric on M there exists a Riemannian metric with the same connection. We have proved that (M, F) is a Berwald space. Therefore, there exists a Riemannian metric g_1 on M with the same connection as F. In [10], weshowed that the connection of a globally (geodesic) symmetric Berwald space is affine symmetric. Therefore, (M, g_1) is a Riemannian globally symmetric space ([13]). ∎

From the proof of Theorem 2.7, we obtain the following

COROLLARY 2.8: *Let $(G/K, F)$ be a globally symmetric Finsler space and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding decomposition of the Lie algebras. Let π be the natural mapping of G onto G/K. Then $(d\pi)_e$ maps $\mathfrak{p}$ isomorphically onto the tangent space of G/K at $p_0 = eK$. If $X \in \mathfrak{p}$, then the geodesic emanating from p_0 with initial tangent vector $(d\pi)_e X$ is given by*

$$\gamma_{d\pi \cdot X}(t) = \mathrm{exp}tX \cdot p_0.$$

Furthermore, if $Y \in T_{p_0}(G/K)$, then $(d\mathrm{exp}tX)_{p_0}(Y)$ is the parallel of Y along the geodesic.

3. Minkowski symmetric Lie algebras

In this section, we use the notion of Minkwoski symmetric Lie algebras to give an algebraic description of symmetric Finsler spaces. We first give the definition (cf. [11]).

Definition 3.1: Let $(\mathfrak{g}, \sigma)$ be a symmetric Lie algebra and $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the canonical decomposition of $\mathfrak{g}$ with respect to the involution σ. If F is a Minkowski norm on $\mathfrak{m}$ and the following condition is satisfied

$$g_y([x, u], v) + g_y(u, [x, v]) + 2C_y([x, y], u, v) = 0,$$

$\forall y \neq 0, y, u, v \in \mathfrak{m}$, $x \in \mathfrak{k}$, where g_y and C_y are the fundamental form and Cartan torsion of F at y, respectively. Then $(\mathfrak{g}, \sigma, F)$ is called a Minkowski symmetric Lie algebra.

Using the notion of Minkowski symmetric Lie algebras, we can give an algebraic description of globally symmetric Finsler spaces. In the following, if (G, K) is a symmetric pair, we use σ to denote the involutive automorphism of G as well as that of the Lie algebra $\mathfrak{g}$ of G. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the canonical decomposition of $\mathfrak{g}$ with respect to σ. As usual, we identify the tangent space $T_o(G/K)$ with $\mathfrak{p}$. Since the proof of the following theorem is almost the same as in [11], where we consider Berwald spaces, we omit it.

THEOREM 3.1: *Let $(G/K, F)$ be a globally symmetric Finsler space. Then $(\mathfrak{g}, \sigma, F)$ is a Minkowski symmetric Lie algebra. Conversely, Let $(\mathfrak{g}, \sigma, F)$ be a*

Minkowski symmetric Lie algebra and suppose (G, K) is any pair associated with $(\mathfrak{g}, \sigma)$ such that K is closed and connected. Then there exists a Finsler metric (still denoted by F) on G/K such that $(G/K, F)$ is a locally symmetric Finsler space. Furthermore, if the pair (G, K) is a symmetric pair, then G/K with this metric is globally symmetric.

As pointed out in [11], if $(\mathfrak{g}, \sigma, F)$ is a Minkowski symmetric Lie algebra, then $(\mathfrak{g}, \sigma)$ is an orthogonal symmetric Lie algebra. We call $(\mathfrak{g}, \sigma, F)$ of the Euclidean type, the compact type or the noncompact type if $(\mathfrak{g}, \sigma)$ is of Euclidean, compact or noncompact type, respectively. Accordingly, a globally symmetric Finsler space is called of the Euclidean, the compact or the noncompact type if the associated Minkowski symmetric Lie algebra is of the corresponding type.

4. The duality

By Theorem 2.7, the duality of symmetric Finsler spaces is the same as that of the symmetric Berwald spaces. Therefore, we only need to restate the results of [11].

PROPOSITION 4.1: *Let $(\mathfrak{g}, \sigma, F)$ be a Minkowski symmetric Lie algebra and $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be the canonical decomposition. Let $\mathfrak{g}^* = \mathfrak{h} + \sqrt{-1}\mathfrak{m}$ be the real subalgebra of $(\mathfrak{g}^C)_R$. Define a Minkowski norm on $\sqrt{-1}\mathfrak{m}$ by*

$$F^*(\sqrt{-1}u) = F(u), \quad u \in \mathfrak{m};$$

and let σ^* be the involutive automorphism of $\mathfrak{g}^*$ induced by σ. Then $(\mathfrak{g}^*, \sigma^*, F^*)$ is a Minkowski symmetric Lie algebra. Moreover, if $(\mathfrak{g}, \sigma, F)$ is of the compact resp. the noncompact type, then $(\mathfrak{u}, \sigma^*, F^*)$ is of the noncompact resp. the compact type and conversely.

Definition 4.1: The Minkowski symmetric Lie algebra $(\mathfrak{g}^*, \sigma^*, F^*)$ defined in Proposition 4.1 is called the dual of $(\mathfrak{g}, \sigma, F)$.

It is clear that, if $(\mathfrak{g}_1, \sigma_1, F_1)$ is the dual of $(\mathfrak{g}_2, \sigma_2, F_2)$, then $(\mathfrak{g}_2, \sigma_2, F_2)$ is the dual of $(\mathfrak{g}_1, \sigma_1, F_1)$.

Definition 4.2: Let $(\mathfrak{g}_i, \sigma_i, F_i)$, $i = 1, 2$, be two Minkowski symmetric Lie algebras. Then they are called isomorphic if there exists an (Lie algebra) isomorphism φ of $\mathfrak{g}_1$ onto $\mathfrak{g}_2$ such that $\varphi \circ \sigma_1 = \sigma_2 \circ \varphi$ and $F_1(x) = F_2(\varphi(x))$, $\forall x \in \mathfrak{g}_1$.

PROPOSITION 4.2: *Let $(\mathfrak{g}_i, \sigma_i, F_i)$ be two Minkowski symmetric Lie algebras. Then $(\mathfrak{g}_1, \sigma_1, F_1)$ is isomorphic to $(\mathfrak{g}_2, \sigma_2, F_2)$ if and only if $(\mathfrak{g}_1^*, \sigma_1^*, F_1^*)$ is isomorphic to $(\mathfrak{g}_2^*, \sigma_2^*, F_2^*)$.*

5. Decomposition theorems

In the Riemannian case, every (simply connected) Riemannian symmetric space can be uniquely decomposed into the product of an Euclidean space, a Riemannian symmetric space of the compact type and a Riemannian space of the noncompact type. Furthermore, each Riemannian space of the compact type or the noncompact type can be uniquely decomposed into the product of the irreducible ones (cf. [13]). It is natural to consider the same problem for symmetric Finsler spaces. However, we must be careful because in this case we do not have the orthogonality. Therefore, the product of two symmetric Finsler spaces may not be unique. To give the accurate definition, we need the following result:

THEOREM 5.1: *Let (M, F) be a globally symmetric Finsler space and (G, K) an effective Riemannian symmetric pair such that $M \simeq G/K$. Then the connection of F (as a Berwald metric) coincides with the canonical connection of G/K. In particular, for any effective Riemannian symmetric pair (G, K), the connection is the same for all G-invariant Finsler metrics on G/K.*

Proof: By Theorem 2.7, the connection of F coincides with the Levi-Civita connection of a G-invariant Riemannian metric g on G/K. But the Levi-Civita connection on G/K is the same for all G-invariant Riemannian metrics on G/K (this is called the canonical connection, see [13] and [14]). Therefore, the theorem follows. ∎

By Theorem 5.1, to consider the decomposition of symmetric Finsler spaces, we need not consider the connection. Therefore, we have the following

Definition 5.1: Let (M, F), (M_1, F_1), (M_2, F_2) be globally symmetric Finsler spaces. Then (M, F) is called the product of (M, F_1) and (M_2, F_2) if $M \simeq M_1 \times M_2$ and $F_i = F|_{M_i}$, $i = 1, 2$. A globally symmetric Finsler space of the compact or the noncompact type is called irreducible if it can not be written as the product of two globally symmetric Finsler spaces.

Similarly, we can define the notion of product of finitely many globally symmetric Finsler spaces. Now we have

THEOREM 5.2: *Let (M, F) be a simply connected globally symmetric Finsler space. Then (M, F) can be decomposed into the product of a reversible Minkowski space, a globally symmetric Finsler space of the compact type and a globally symmetric Finsler space of the noncompact type. Moreover, every simply connected symmetric Finsler space of compact or noncompact type can be decomposed into the product of irreducible symmetric Finsler spaces. The decomposition is unique as manifolds (in general not unique as Finsler spaces).*

Proof: By Theorem 2.7, the only thing we need to check is that the coset space $\mathbb{R}^n \simeq \mathbb{R}^n/\{0\}$ endowed with a Finsler metric invariant under the parallel translation is a Minkowski space. But this is obvious. ∎

Remark: Although we have the decomposition theorem, the classification of symmetric Finsler spaces is not reduced to the case of irreducible ones, because the product of two or more symmetric Finsler spaces is not unique.

6. The flag curvature

In [11], we give the formula for the flag curvature of symmetric Berwald spaces. By Theorem 2.7, it is also valid for globally symmetric Finsler spaces. We only state the results here.

THEOREM 6.1: *Let $(\mathfrak{g}, \sigma, F_0)$ be a Minkowski symmetric Lie algebra. Let (G, K) be a pair associated with $(\mathfrak{g}, \sigma)$, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the canonical decomposition of $(\mathfrak{g}, \sigma)$ and suppose there exists an invariant Finsler metric F on G/K (this is the case when K is connected and closed, see [11]) such that the restriction of F to $\mathfrak{p}$ is F_0. Then the curvature tensor of F is given by:*

$$R_o(u, v)w = -[[u, v], w], \quad \forall u, v, w \in \mathfrak{p},$$

and the flag curvature of the flag (P, y), $y \neq 0, y \in P$ is given by

$$K(P, y) = g_l([[l, v], l], v), \tag{6.1}$$

where $l = y/F$ (the distinguished section) and $\{l, u\}$ is an orthonormal basis of the plane P with respect to g_l.

COROLLARY 6.2: *Let $(\mathfrak{g}, \sigma, F_0)$ be a Minkowski symmetric Lie algebra and $(\mathfrak{g}^*, \sigma^*, F_0^*)$ be its dual. Let (G, K) and (G^*, K^*) be two pairs associated with $(\mathfrak{g}, \sigma, F_0)$ and $(\mathfrak{g}^*, \sigma^*, F_0^*)$, respectively. Suppose there exist invariant Finsler metrics F on G/K and F^* on G^*/K^* such that the restriction of F and F^**

on $\mathfrak{p}$ *and* $\sqrt{-1}\mathfrak{p}$ *is equal to* F_0 *and* F_0^**, respectively. Then the flag curvature of* $(G/K, F)$ *and* $(G^*/K^*, F^*)$ *satisfies:*

$$K(P, y) = -K(\sqrt{-1}P, \sqrt{-1}y), \quad 0 \neq y \in \mathfrak{p},$$

where P *is a plane in* $\mathfrak{p}$ *containing* y *and* $\sqrt{-1}P$ *is the plane in* $\sqrt{-1}\mathfrak{p}$ *spanned by* $\sqrt{-1}u, u \in P$.

THEOREM 6.3: *Let* $(G/H, F)$ *be globally symmetric Finsler space with* G *semisimple and* $(\mathfrak{g}, \sigma, F)$ *the associated Minkowski symmetric Lie algebra with the canonical decomposition* $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$. *Then*

(1) *if* $(\mathfrak{g}, \sigma)$ *is of the compact type, then the flag curvature of* $(G/H, F)$ *is everywhere* ≥ 0*;*
(2) *if* $(\mathfrak{g}, \sigma)$ *is of the noncompact type, then the flag curvature of* $(G/H, F)$ *is everywhere* ≤ 0.

Moreover, in (1) and (2), the flag curvature vanishes if and only if in the flag (P, y)*, the plane* P *is commutative.*

7. The Ricci Scalar

By Theorem 6.1, to compute the Ricci scalar of the symmetric Finsler space $(G/K, F)$, we only need to compute the trace of the endomorphism $-(ad\, y)^2$ of the vector space $\mathfrak{p}$. The computation will be completed through several lemmas.

Since (G, K) is a Riemannian symmetric pair, $(\mathfrak{g}, \sigma)$ is an orthogonal symmetric Lie algebra. By the theory of orthogonal Lie algebras ([13]), $\mathfrak{g}$ has the decomposition as a direct sum of ideals:

$$\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2, \tag{7.1}$$

where $\mathfrak{g}_0$ is an abelian Lie algebra, $\mathfrak{g}_1$ ($\mathfrak{g}_2$) is a noncompact (compact) semisimple Lie algebra. Moreover, let σ_i, $i = 0, 1, 2$ be the restriction of σ to $\mathfrak{g}_i$, then $(\mathfrak{g}, \sigma_0), (\mathfrak{g}_1, \sigma_1), (\mathfrak{g}_2, \sigma_2)$ are orthogonal symmetric Lie algebras of the Euclidean, the noncompact and the compact type, respectively. Let $\mathfrak{g}_i = \mathfrak{k}_i + \mathfrak{p}_i$ be the canonical decomposition of the orthogonal symmetric Lie algebra $(\mathfrak{g}_i, \sigma_i)$, $i = 0, 1, 2$.

LEMMA 7.1: *Let* $y = y_0 + y_1 + y_2$ *be the decomposition of* y *corresponding to* (7.1). *Then* $Ric(y) = Ric(y_0) + Ric(y_1) + Ric(y_2)$. *Furthermore,* $Ric(y_i)$ *is equal to the Ricci scalar of* G_i/K_i *at* y_i, $i = 0, 1, 2$.

Proof: Since (7.1) is the direct sum of ideals, we have $[y_0, x] \in \mathfrak{g}_0$, $\forall x \in \mathfrak{p}$. Hence $[y_1, [y_0, x]] \in \mathfrak{g}_0 \cap \mathfrak{g}_1 = 0$. This means that $(ad\, y_1)(ad\, y_0) = 0$ as an endomorphism of $\mathfrak{p}$. Similarly we have $(ad\, y_i)(ad\, y_j) = 0$, $i \neq j$, $i, j = 0, 1, 2$. Therefore, we have (as endomorphisms of $\mathfrak{p}$)

$$(ad\, y)^2 = (ad\, y_0)^2 + (ad\, y_1)^2 + (ad\, y_2)^2.$$

Taking the trace in the above equation we get $Ric(y) = Ric(y_0) + Ric(y_1) + Ric(y_2)$. Since $(ady_i)(\mathfrak{p}_j) = 0$, for $i \neq j$, we have

$$Tr((ady_i)^2|_{\mathfrak{p}}) = Tr((ady_i)^2|_{\mathfrak{p}_i}).$$

This proves the lemma. ∎

It is obvious that $Ric(y_0) = 0$. Therefore, Lemma 7.1 reduces the computation to the cases of symmetric Finsler space of the compact and the noncompact type. By the duality, we only need to consider the compact type. Therefore, in the following, we assume that G/K is a symmetric Finsler space of the compact type. Then $\mathfrak{g}$ is a compact semisimple Lie algebra.

We still need some algebraic preliminaries. Details can be found in [13] (Chapter VII). Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ containing y. Extend a to a maximal abelian subspace $\mathfrak{t}$ of $\mathfrak{g}$. Then the complexification $\mathfrak{t}^C$ is a Cartan subalgebra of $\mathfrak{g}^C$. Let Δ be the root system of $\mathfrak{g}^C$ with respect to $\mathfrak{t}^C$ and $\Delta_{\mathfrak{p}}$ be the subset of roots in Δ which do not vanish identically on $\mathfrak{a}^C$. Let Σ be the corresponding set of restricted roots. Fix certain compatible ordering in the sense of [13]. Let Δ^+ and Σ^+ be the set of positive roots and positive restricted roots, respectively. For any linear form λ in $\mathfrak{a}^C$, put

$$\mathfrak{k}_\lambda = \{w \in \mathfrak{k} |\, (ad\, x)^2 w = \lambda(x)^2 w, \forall x \in \mathfrak{a}\},$$
$$\mathfrak{p}_\lambda = \{z \in \mathfrak{p} |\, (ad\, x)^2 z = \lambda(x)^2 z, \forall x \in \mathfrak{a}\}.$$

Then $\mathfrak{k}_{-\lambda} = \mathfrak{k}_\lambda$, $\mathfrak{p}_{-\lambda} = \mathfrak{p}_\lambda$. It is proved in [13] that

$$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\lambda \in \Sigma^+} \mathfrak{k}_\lambda, \ \mathfrak{p} = \mathfrak{a} + \sum_{\lambda \in \Sigma^+} \mathfrak{p}_\lambda. \tag{7.2}$$

Note that since $\mathfrak{g}$ is a compact semisimple Lie algebra, for any $\lambda \in \Sigma$ and $x \in \mathfrak{a}$, $\lambda(x)$ is pure imaginary. Therefore, $\lambda(x)^2$ is a nonpositive real number.

Lemma 7.2: *For any $\lambda \in \Sigma^+$, $\mathfrak{k}_\lambda$ is isomorphic to $\mathfrak{p}_\lambda$ as a vector space.*

Proof: First note that $[\mathfrak{a}, \mathfrak{p}_\lambda] \subset \mathfrak{k}_\lambda$. In fact, for any $x, x' \in \mathfrak{a}$, $z \in \mathfrak{p}_\lambda$, we have

$$\begin{aligned}[x,[x,[x',z]]] &= [x,[[x,x'],z]] + [x,[x',[x,z]]] \\ &= [x,[x',[x,z]]] \\ &= [[x,x'],[x,z]] + [x',[x,[x,z]]] \\ &= [x',[x,[x,z]]] \\ &= [x', \lambda(x)^2 z] = \lambda(x)^2[x',z].\end{aligned}$$

This proves our assertion. Similarly we have $[\mathfrak{a}, \mathfrak{k}_\lambda] \subset \mathfrak{p}_\lambda$. Since $\lambda \in \Sigma$, we can select $x_0 \in \mathfrak{a}$ such that $\lambda(x_0) \neq 0$. Then it is obvious that $ad\, x_0$ is a one-to-one linear mapping from $\mathfrak{k}_\lambda$ to $\mathfrak{p}_\lambda$ as well as from $\mathfrak{p}_\lambda$ to $\mathfrak{k}_\lambda$. Therefore, $\mathfrak{k}_\lambda$ is linearly isomorphic to $\mathfrak{p}_\lambda$. ∎

With the above preparation, we now obtain the formula for the Ricci scalar for symmetric Finsler spaces of the compact type.

LEMMA 7.3: *If G/K is of the compact type, then $Ric(y) = -1/2B(y,y)$, where B is the Killing form of the Lie algebra $\mathfrak{g}$.*

Proof: Let Σ^* be the subset of Σ^+ consisting of the elements which takes nonzero value on y. By (7.2) we have

$$Tr((ad\, y)^2|_{\mathfrak{p}}) = \sum_{\lambda \in \Sigma^*} \lambda(y)^2 \dim \mathfrak{p}_\lambda,$$

$$Tr((ad\, y)^2|_{\mathfrak{k}}) = \sum_{\lambda \in \Sigma^*} \lambda(y)^2 \dim \mathfrak{k}_\lambda.$$

By Lemma 7.2, we have $\dim \mathfrak{k}_\lambda = \dim \mathfrak{p}_\lambda$, $\forall \lambda \in \Sigma$. Therefore, we have

$$Tr((ad\, y)^2|_{\mathfrak{p}}) = Tr((ad\, y)^2|_{\mathfrak{k}}).$$

Since $\mathfrak{k}$ and $\mathfrak{p}$ are invariant subspaces of $(ad\, y)^2$, we have

$$\begin{aligned}B(y,y) = Tr((ad\, y)^2|_{\mathfrak{g}}) &= Tr((ad\, y)^2|_{\mathfrak{k}}) + Tr((ad\, y)^2|_{\mathfrak{p}}) \\ &= 2Tr((ad\, y)^2|_{\mathfrak{p}}).\end{aligned}$$

Therefore,

$$Ric(y) = -Tr((ad\, y)^2|_{\mathfrak{p}}) = -\frac{1}{2}B(y,y). \qquad ∎$$

By the duality, Lemma 7.3 is also valid for a symmetric Finsler space of the non compact type. In summarizing, we have proved

THEOREM 7.4: *Let $(G/K, F)$ be a symmetric Finsler space and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the canonical decomposition of the corresponding Minkowski symmetric Lie algebras. Identifying $\mathfrak{p}$ with the tangent space $T_o(G/K)$ at the origin, we have $Ric(y) = -\frac{1}{2}B(y, y)$, $\forall y(\neq 0) \in \mathfrak{p}$, where B denotes the Killing form of the Lie algebra $\mathfrak{g}$.*

As an application of the formula of the Ricci scalar, we have

THEOREM 7.5: *Let $(G/H, F)$ be a n-dimensional symmetric Finsler space. Suppose F is an Einstein metric, i.e., $Ric(y) = (n-1)cF(y)^2$, for any nonzero $y \in T(G/H)$, where c is a constant. Then the following assertions hold:*

1) *if $c = 0$, then F is locally Minkowskian;*
2) *if $c > 0$, then F is Riemannian and G/H is compact;*
3) *if $c < 0$, then F is Riemannian and G/H is diffeomorphic to a Euclidean space.*

Proof: Let $(\mathfrak{g}, \sigma)$ be the orthogonal symmetric Lie algebra associated with the Riemannian symmetric pair (G, K) and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the canonical decomposition. As above, we identify $\mathfrak{p}$ with the tangent space $T_o(G/K)$. Let $\mathfrak{g}$ be decomposed into the direct sum of ideals as in (7.1). Recall that if $\mathfrak{g}$ is semisimple, then the restriction of the Killing form to $\mathfrak{p}$ is positive definite or negative definite according to it being noncompact or compact ([13]). Now we prove the theorem case by case.

1) If $c = 0$, then for any $y \in \mathfrak{p}$ we have

$$0 = Ric(y) = -\frac{1}{2}B(y, y).$$

This means that in the decomposition (7.1), we have $\mathfrak{g}_1 = \mathfrak{g}_2 = 0$. Therefore, $(\mathfrak{g}, \sigma)$ is of the Euclidean type. Hence F is locally Minkowskian.

2) If $c > 0$, then we have,

$$Ric(y) = cF(y)^2 = -\frac{1}{2}B(y, y) > 0, \quad \forall y(\neq 0) \in \mathfrak{p}.$$

This means that the Killing form is negative definite on $\mathfrak{p}$. Hence in (7.1) we have $\mathfrak{g}_0 = \mathfrak{g}_1 = 0$. That is, $\mathfrak{g}$ is compact semisimple. Thus G is a compact semisimple Lie group. Therefore, G/K is compact. On the other hand, since

$$F(y) = \sqrt{-\frac{1}{2c}B(y, y)}, \quad \forall y \neq 0,$$

and $-\frac{1}{2c}B(y, y)$ is an inner product on $\mathfrak{p}$, F is Riemannian.

3) Suppose that $c < 0$. Then similarly as in case 2), we can prove that F is Riemannian and G is a noncompact semisimple Lie group. Thus $(\mathfrak{g}, \sigma)$ is an orthogonal symmetric Lie algebra of the noncompact type. Therefore, G/K is diffeomorphic to a Euclidean space ([13]). ∎

8. Locally symmetric Finsler spaces and some rigidity results results

In this section, we will give a geometric description of locally symmetric Finsler spaces. We first note a result of Busemeann and Phadke ([6]) which asserts that a locally symmetric G-space has a globally symmetric universal covering space. When the G-space is smooth, it is a (complete) Finsler space. Therefore, combining Theorem 2.2 (d) and Theorem 2.7, we have:

THEOREM 8.1: *Let (M, F) be a complete locally symmetric Finsler space. Then for any $x \in M$, there exists a simply connected globally symmetric Finsler space $(\tilde{M}, \tilde{F})$, a neighborhood N of x in M, and an isometry of N onto an open subset $\tilde{N}$ of $\tilde{M}$. In particular, (M, F) is a Berwald space.*

In [10], we proved that a Berwald space is locally symmetric if and only if the flag curvature is invariant under the parallel displacements of its linear connection. Thus we have

THEOREM 8.2: *Let (M, F) be a complete Finsler space. Then (M, F) is locally symmetric if and only if it is Berwaldian and the flag curvature is invariant under all parallel displacements.*

The results of this paper imply some global rigidity results related to the flag curvature in Finsler manifolds. First we have

THEOREM 8.3: *Let (M, F) be a complete locally symmetric Finsler space. If the flag curvature of (M, F) is everywhere nonzero, then F is Riemannian.*

Proof: Let $x \in M$. By Theorem 8.1, there exists a neighborhood N of x, a simply connected globally symmetric Finsler space $(\tilde{M}, \tilde{F})$ and an isometry from N onto an open subset $\tilde{N}$ of $\tilde{M}$. By the assumption, the flag curvature of $(\tilde{M}, \tilde{F})$ is everywhere nonzero on $\tilde{N}$. Since $(\tilde{M}, \tilde{F})$ is homogeneous (Theorem 2.2 (c)), we see that the flag curvature of $(\tilde{M}, \tilde{F})$ is everywhere nonzero. By Theorem 2.4, $\tilde{M}$ can be written as a coset space of a Riemannian symmetric pair (G, K) and $\tilde{F}$ corresponds to a G-invariant Finsler metric $\tilde{F}_1$ on G/K. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the canonical decomposition of the Lie algebra. Then by the formula (6.1), we see that $[u, v] \neq 0$, for any vectors u, v in $\mathfrak{p}$ which are linearly independent. That is, G/K is irreducible of rank one. Fix a G-invariant Riemannian metric $\tilde{g}$ on $\tilde{G}/\tilde{K}$. Then $Ad(K)$ acts transitively on the unit sphere of the Euclidean space $(T_o(\tilde{G}/\tilde{K}), \tilde{g})$. Since both $\tilde{g}$ and $\tilde{F}_1$ are invariant under the action of $Ad(K)$, this implies that there exists a positive number c such that $\tilde{F}_1(y) = c\sqrt{\tilde{g}(y,y)}$, for any $y \in T_o(\tilde{G}/\tilde{K})$. Hence, $\tilde{F}_1$ is Riemannian. Thus $\tilde{F}$ is Riemannian. Consequently, F is Riemannian on N. Since x is arbitrary, we conclude that F is a Riemannian metric. ∎

Since a compact Finsler space is complete ([4]), we have

Corollary 8.4: *Let (M, F) be a compact locally symmetric Finsler space. If the flag curvature of (M, F) is everywhere nonzero, then F is Riemannian.*

It should be noted that the negatively curved case in Corollary 8.4 is also a corollary of the main result in [12]. It is still not clear whether the results hold if we drop the complete assumption in Theorems 8.1 through Theorem 8.3. But we have

Theorem 8.5: *Let (M, F) be a locally symmetric Berwald space. If the flag curvature of (M, F) is everywhere nonzero, then F is Riemannian.*

Proof: In our previous paper [10], we proved that a locally symmetric Berwald space is locally isometric to a globally symmetric Berwald space. Therefore, the result can be proved similarly as Theorem 8.3. ∎

By the result of [10], we easily see that a Berwald space of constant flag curvature must be locally symmetric. Therefore, we have

Corollary 8.6: *A Berwald space of nonzero constant flag curvature is Riemannian.*

Note that this result is not new. It is a special case of the Numata's theorem (see [4]). But the approaches are very different.

Finally, we make a conjecture about locally symmetric Finsler spaces to conclude this paper.

Conjecture: *A locally symmetric (not necessarily complete) Finsler space is a Berwald space.*

If this conjecture is true, then the results of Theorems 8.1 through 8.3 still hold without the complete assumption.

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